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ON THE POSSIBILITY OF USING HARMONIC FUNCTIONS FOR SOLVING PROBLEMS OF THE THEORY OF ELASTICITY OF NONHOMOGENEOUS MEDIA

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We consider a nonhomogeneous isotropic medium, whose shear modulus is a power of a linear binomial in the Cartesian coordinates while Poisson's ratio is constant. The conditions are found under which the general solution of the plane and three-dimensional problems of the theory of elasticity can be expressed in terms of harmonic functions. Also some special cases of the variation of the shear modulus for a variable Poisson's ratio are considered. The obtained results are used for solving the problem of the stress-strain state of a nonhomogeneous half-space under the action of concentrated forces, applied normally and tangentially to the boundary surface.

1. The general solution of the three-dimensional problem of the theory of elasticity for a nonhomogeneous isotropic medium, whose shear modulus G and Poisson's ratio ν are differentiable functions of the coordinate z, has the form [1]

$$u_x = -\frac{1}{2G}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial x} + \frac{\partial N}{\partial y}$$

$$u_y = -\frac{1}{2G}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial y} - \frac{\partial N}{\partial x} \tag{1.1}$$

$$u_z = -\frac{1}{G}\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial z} + \frac{\partial}{\partial z}\left[\frac{1}{2G}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)L\right]$$

Here u_x, u_y, u_z are the components of the displacement vector, ∇^2 is the three-dimensional Laplace operator and L, N are functions satisfying the equations

$$\nabla^2\nabla^2 L - \frac{G}{1-\nu}\left\{\frac{1}{G}\left[\frac{\partial^2}{\partial z^2}(\nu\nabla^2 L) - \nu\frac{\partial^2}{\partial z^2}\nabla^2 L\right] - \right.$$

$$\left. - 2\frac{\partial}{\partial z}\left[(1-\nu)\nabla^2 L\right]\frac{d}{dz}\left(\frac{1}{G}\right) + \left(\nu\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\frac{d^2}{dz^2}\left(\frac{1}{G}\right)\right\} = 0 \tag{1.2}$$

$$\nabla^2 N + q(z)\frac{\partial N}{\partial z} = 0 \qquad \left(q(z) = \frac{d}{dz}\ln G\right) \tag{1.3}$$

The general solution of the axisymmetric problem is expressed in terms of the function L. We can arrive at the two-dimensional problem of the theory of elasticity of nonhomogeneous media if we put $N = 0$ and the function L does not depend on the coordinates x or y. In addition, ν has to be replaced by ν^*, where $\nu^* = \nu$ in the case of the plane strain and $\nu^* = \nu/(1+\nu)$ in the case of the plane state of stress.

Thus, the solving of any problem of the theory of elasticity reduces to the finding, from Eqs. (1.2) and (1.3), of the functions L and N satisfying the given boundary conditions. In general, in the solving of these equations we encounter great difficulties which reduce to a significant degree the practical value of the above given results. The formulation and the solving of concrete problems is substantially simplified only if, in expressing the general solution, we succeed to use classes of functions which, relatively, have been well investigated. We now examine the possibility of representing the general solution of Eqs. (1.2) and (1.3) in terms of harmonic functions in the case when the shear modulus varies according to a power relation of the form

$$G(z) = G_0 (1 + cz)^b \tag{1.4}$$

First we consider Eq. (1.2), which we write in a more compact form

$$\nabla^2\left(\frac{1-\nu}{G}\nabla^2 L\right) - \left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) L \frac{d^2}{dz^2}\left(\frac{1}{G}\right) = 0 \tag{1.5}$$

2. We assume that Poisson's ratio is an arbitrary function of the coordinate z, while the shear modulus

$$G(z) = G_0/(1 + cz) \tag{2.1}$$

Then Eq. (1.5) can be simplified and its general solution can be obtained from the Poisson equation

$$\Delta^2 L = \chi(z)\varphi_0, \quad \chi(z) = G / (1 - \nu) \tag{2.2}$$

Here and in the sequel, φ_j $(j = 0, 1, 2, \ldots)$ are arbitrary harmonic functions. We consider the harmonic function φ_2, related to φ_0 by

$$\varphi_0 = 2d\varphi_2/dz$$

By straighforward verification we can see that a particular solution of Eq. (2.2) can be taken in the form

$$L^* = \int_{t_0}^{z} \chi(t)\,[\varphi_2(x, y, z) - \varphi_2(x, y, 2t - z)]\,dt \tag{2.3}$$

Hence it follows that

$$L = \varphi_1 + L^* \tag{2.4}$$

Thus, in the case when the shear modulus varies according to (2.1), while Poisson's ratio $\nu = \nu(z)$, the general solution of Eq. (1.5) can be expressed in terms of two arbitrary harmonic functions.

Hence, in particular, it follows that two harmonic functions are sufficient for the solution of any plane or axisymmetric problem if the body under consideration possesses the indicated nonhomogeneity.

3. We assume that in Eq. (1.5) we have $\nu = \text{const}$, while the shear modulus is a power function of the coordinate z of the form (1.4).

We will seek L in the form

$$L = (1 + cz)^\beta \sum_{k=0}^{\infty} a_k (1 + cz)^{-k} \psi_k \tag{3.1}$$

where ψ_k $(k = 0, 1, 2 \ldots)$ are arbitrary harmonic functions connected through the relations

$$\psi_k = \partial\psi_{k+1} / \partial z \tag{3.2}$$

Substituting L into Eq. (1.5) and collecting similar terms, we equate to zero the coefficients of like powers of $1 + cz$. From the obtained algebraic system it follows that the formal solutions of (1.5) are two series

$$L_1 = (1 + cz)^{\beta_1} \sum_{k=0}^{\infty} a_k^{(1)} (1 + cz)^{-k} \psi_k^{(1)} \tag{3.3}$$

$$L_2 = (1 + cz)^{\beta_2} \sum_{k=0}^{\infty} a_k^{(2)} (1 + cz)^{-k} \psi_k^{(2)} \tag{3.4}$$

$$\beta_1 = {}^1/_2 (b + 1 + \xi), \qquad \beta_2 = {}^1/_2 (b + 1 - \xi)$$

$$\xi = \sqrt{(b + 1) [1 - \nu b / (1 - \nu)]}$$

$$a_k^{(1)} = \frac{(\xi - 2k + 1)^2 - (b + 2)^2}{8k} c a_{k-1}^{(1)}, \qquad a_0^{(1)} = 1$$

$$a_k^{(2)} = \frac{(\xi + 2k - 1)^2 - (b + 2)^2}{8k} c a_{k-1}^{(2)}, \qquad a_0^{(2)} = 1 \tag{3.5}$$

The indices 1 and 2 for the harmonic functions ψ_k have been introduced in order to emphasize that these functions are, in general, distinct. In certain particular cases of nonhomogeneity of the elastic medium, the series (3.3) and (3.4) terminate and then the general solution of Eq. (1.5) can be expressed in the form of a finite sum. To find these cases, we set

$$a_{n+1}^{(1)} = a_{m+1}^{(2)} = 0 \tag{3.6}$$

i.e. the first series breaks at the term with index n, while the second one at the term with index m. From the expressions (3.5) it follows that equality (3.6) is possible only under the condition

$$(\xi - 2n - 1)^2 - (b + 2)^2 = 0, \quad (\xi + 2m + 1)^2 - (b + 2)^2 = 0 \tag{3.7}$$

Hence, taking into account that $\xi \geqslant 0$ and $0 \leqslant \nu \leqslant {}^1/_2$, we find

$$b = n + m - 1, \quad \xi = n - m, \quad \nu = \frac{n + m - (n - m)^2}{4nm} \tag{3.8}$$

and for n and m we obtain the system of inequalities

$$n \geqslant m \geqslant 1, \qquad n \leqslant m + {}^1/_2 [1 + \sqrt{8m + 1}] \tag{3.9}$$

The function L can be represented in the form

$$L = \sum_{k=0}^{k=n} a_k^{(1)} (1 + cz)^{n-k} \frac{\partial^{n-k}\varphi_1}{\partial z^{n-k}} + \sum_{k=0}^{k=m} a_k^{(2)} (1 + cz)^{m-k} \frac{\partial^{m-k}\varphi_2}{\partial z^{m-k}}$$

$$a_k^{(1)} = \frac{(n - m - 2k + 1)^2 - (n + m + 1)^2}{8k} c a_{k-1}^{(1)}, \qquad a_0^{(1)} = 1 \tag{3.10}$$

$$a_k^{(2)} = \frac{(n - m + 2k - 1)^2 - (n + m + 1)^2}{8k} c a_{k-1}^{(2)}, \qquad a_0^{(2)} = 1$$

Where $\varphi_1 = \psi_n^{(1)}$ and $\varphi_2 = \psi_m^{(2)}$. Thus, for special cases of nonhomogeneity of the elastic medium, defined by the relations (3.8), the general solution of (1.5) can be represented in terms of two harmonic functions φ_1 and φ_2 in the form (3.10).

Table 1.

b	ν	m	n
2	$^1/_4$	1	2
3	0	1	3
4	$^1/_6$	2	3
5	$^1/_{16}$	2	4
6	$^1/_8$	3	4
7	$^1/_{15}$	3	5
8	0	3	6
8	$^1/_{10}$	4	5
9	$^1/_{16}$	4	6
10	$^1/_{56}$	4	7
10	$^1/_{12}$	5	6

In the table we have indicated for all possible values of $b \leqslant 10$ the corresponding values of the quantities m, n and of the Poisson ratio ν, computed with the aid of the formulas (3.8), (3.9). Because of its small practical value, we have not included in it the case $m = n$, since then $L_1 = L_2$.

4. We consider now Eq. (1.3) with $G(z)$ given by a relation of the form (1.4). Poisson's ratio does not occur in the

equation, therefore we will consider it as an arbitrary function of the coordinate z.

By analogy with the previous case we seek the function N in the form

$$N = (1 + cz)^{\gamma} \sum_{k=0}^{\infty} d_k (1 + cz)^{-k} \psi_k \tag{4.1}$$

From Eq. (1.3) we obtain

$$\gamma = -\frac{b}{2}, \quad d_k = \frac{(2k-1)^2 - (b-1)^2}{8k} c d_{k-1}, \quad d_0 = 1 \tag{4.2}$$

Consequently, the series (4.1) breaks at the term of index s if

$$(2s + 1)^2 - (b - 1)^2 = 0 \tag{4.3}$$

This equality is possible only if

$$b = 2(s + 1) \quad \text{or} \quad b = -2s \qquad (s = 0, 1, 2 \ldots) \tag{4.4}$$

The function N takes the form

$$N = (1 + cz)^{-b/2} \sum_{k=0}^{k=s} d_k (1 + cz)^{-k} \frac{\partial^{s-k} \varphi_3}{\partial z^{s-k}} \tag{4.5}$$

Here $\varphi_3 = \psi_s$.

Thus, the general solution of Eq. (1.3) can be expressed in terms of an arbitrary harmonic function φ_3, if the exponent b in the formula (1.4) can be represented in the form (4.4).

We consider the special case of nonhomogeneity of the elastic medium when the shear modulus is constant while Poisson's ratio is $\nu = \nu(z)$. To this end, in the formulas (2.4) and (4.5) we set $G = \text{const}$. As a result we obtain that the general solution of the three-dimensional problem can be expressed in terms of three arbitrary harmonic functions, since

$$L = \varphi_1 + \int_{t_0}^{z} [\varphi_2(x, y, z) - \varphi_2(x, y, 2t - z)] \frac{dt}{1 - \nu(t)}, \quad N = \varphi_3 \tag{4.6}$$

The obtained results allow us to solve a series of new problems in the theory of elasticity of nonhomogeneous isotropic media. We consider two of these problems.

5. A concentrated force P, applied at the origin to a nonhomogeneous half-space $z \geqslant 0$, acts in the positive direction of the z-axis. We have to determine the deformation of the half-space if Poisson's ratio is constant while the shear modulus varies with the depth according to a power function (1.4) for those values of the exponent b which are allowed by the relations (3.8) and (4.4).

In [2, 3], devoted to similar problems, the state of stress of the half-space $z \geqslant 0$ with modulus $E(z) = E_0 z^k$ under the action of a concentrated force normal to the surface, was investigated. The physical nonreality of such a medium is obvious, since the elasticity modulus of the half-space at the boundary surface is equal to zero. This circumstance implies, in particular, a limitation on the possible values of the exponent k. Thus, for example, the formulation of the problem on the action of a distributed load makes sense only for $0 \leqslant k < 1$. Therefore it presents interest to investigate the state of stress and strain of a half-space, whose shear modulus is a function of the coordinate z of the from (1.4).

For definiteness, we will consider $b = 2$. For other values of the exponent b the method of solution is similar. From Table 1 and the equalities (4.4) we find

$$\nu = 1/4,\ m = 1,\ n = 2,\ s = 0$$

Hence according to (3.10) and (4.5), we have

$$L = (1+cz)^2 \frac{\partial^2 \varphi_1}{\partial z^2} - 2c(1+cz)\frac{\partial \varphi_1}{\partial z} + \frac{3}{2}c^2\varphi_1 + (1+cz)\frac{\partial \varphi_2}{\partial z} - \frac{3}{2}c\varphi_2 \quad (5.1)$$

$$N = \varphi_3 / (1+cz) \quad (5.2)$$

We take $\varphi_3 = 0$. Then the solution of the problem reduces to the finding of the harmonic functions φ_1 and φ_2 in the domain $z \geqslant 0$, satisfying the given boundary conditions. We seek the functions φ_1 and φ_2 in the form

$$\varphi_1 = \int_0^\infty \frac{A+B}{c^2\alpha^3} e^{-\alpha z} J_0(\alpha r)\, d\alpha \qquad (r = \sqrt{x^2+y^2})$$

$$\varphi_2 = \int_0^\infty \frac{A-B}{c\alpha^3} e^{-\alpha z} J_0(\alpha r)\, d\alpha \quad (5.3)$$

Here $J_0(\alpha r)$ is the zero order Bessel function of the first kind. The arbitrary functions of the parameter α, A and B, must be chosen so that the boundary conditions be satisfied. This means that the normal and shear stresses at the surface of the half-space must be equal to:

$$\sigma_z\big|_{z=0} = -f(r), \quad \tau_{zx}\big|_{z=0} = 0, \quad \tau_{zy}\big|_{z=0} = 0 \quad (5.4)$$

Here $f(r)$ is the applied load, which, in the case of a concentrated force P, acting on the half-space, can be represented in the form [4]

$$f(r) = \frac{P}{2\pi}\int_0^\infty \alpha J_0(\alpha r)\, d\alpha \quad (5.5)$$

Substituting the expressions (5.3) into (5.1), we have

$$L = \int_0^\infty \frac{e^{-\alpha z}}{\alpha^3} J_0(\alpha r)\{A\lambda(1+cz)[\lambda(1+cz)+1] + \quad (5.6)$$

$$+ B[\lambda^2(1+cz)^2 + 3\lambda(1+cz) + 3]\}\, d\alpha \qquad \left(\lambda = \frac{\alpha}{c}\right)$$

Inserting the function L into the formulas (1.1), we obtain the components of the displacement vector and through them, the components of the stress tensor. Substituting the expressions for the determination of the stresses into (5.4), we obtain for the functions A and B a system of algebraic equations, from where we find

$$A = -\frac{P}{2\pi}\frac{\lambda(\lambda+1)}{2\lambda^2+6\lambda+3}, \qquad B = \frac{P}{2\pi}\frac{\lambda^2-\lambda-1}{2\lambda^2+6\lambda+3} \quad (5.7)$$

In the final form the formulas for the displacements are

$$u_r = \frac{cP}{4\pi G_0(1+\zeta)}\int_0^\infty J_1(\lambda\rho)\, e^{-\lambda\zeta}\, \frac{\lambda^2[(2\lambda+1)\zeta-1]}{2\lambda^2+6\lambda+3}\, d\lambda$$

$$u_z = \frac{cP}{4\pi G_0(1+\zeta)}\int_0^\infty J_0(\lambda\rho)\, e^{-\lambda\zeta}\lambda\, \frac{2\lambda^2\zeta + \lambda(\zeta+3)+2}{2\lambda^2+6\lambda+3}\, d\lambda \quad (5.8)$$

$$u_\beta = 0 \qquad (\rho = cr,\ \zeta = cz,\ \beta = \operatorname{arc\,tg} y/x)$$

Here u_r, u_β, u_z are the components of the displacement vector in the cylindrical system of coordinates.

We decompose the rational expressions from the integrands into simple fractions. If we now make use of the formulas [5]

$$\int_0^\infty J_\mu(\lambda\rho)\,d\lambda = \frac{1}{\rho} \qquad (\operatorname{Re}\mu > -1,\ \rho > 0) \tag{5.9}$$

$$\int_0^\infty \frac{\lambda^\mu}{\lambda+\gamma} J_\mu(\lambda\rho)\,d\lambda = \frac{\pi\gamma^\mu}{2\cos\mu\pi} T_{-\mu}(\gamma\rho), \qquad T_{-\mu}(\gamma\rho) = \mathbf{H}_{-\mu}(\gamma\rho) - N_{-\mu}(\gamma\rho)$$

$$(-{}^1/_2 < \operatorname{Re}\mu < {}^3/_2,\ \rho > 0,\ |\arg\gamma| < \pi)$$

where $\mathbf{H}_{-\mu}(\gamma\rho)$ is Struve's function and $N_{-\mu}(\gamma\rho)$ is Neumann's function, then for $\zeta = 0$ we obtain the expressions for the determination of the displacements of the surface points of the nonhomogeneous half-space

$$u_r\big|_{z=0} = -\frac{cP}{32G_0}\left[(3+2\sqrt{3})T_1(\gamma_1\rho) + (3-2\sqrt{3})\,T_1(\gamma_2\rho) - \frac{12}{\pi}\right]$$

$$u_z\big|_{z=0} = \frac{3cP}{8\pi G_0\rho}\left\{1 - \frac{\pi}{3}\rho\,[({}^7/_4+\sqrt{3})\,T_0(\gamma_1\rho) + ({}^7/_4-\sqrt{3})\,T_0(\gamma_2\rho)]\right\} \tag{5.10}$$

$$u_\beta = 0, \quad \gamma_1 = {}^1/_2(3+\sqrt{3}), \quad \gamma_2 = {}^1/_2(3-\sqrt{3})$$

Hence it follows that for $\rho \ll 1$

$$u_r\big|_{z=0} \approx \frac{-P}{8\pi G_0 r}, \quad u_\beta = 0, \quad u_z\big|_{z=0} \approx \frac{3P}{8\pi G_0 r} \tag{5.11}$$

Thus, near the point of application of the force, the displacements of a nonhomogeneous half-space coincide with the displacements of an identically loaded homogeneous half-space having the same Poisson's ratio and with shear modulus equal to G_0. However, as we go farther from the point of application of the force, the displacements die out rapidly. The character of the damping can be easily shown, if we make use of the asymptotic expansion for the function $T_\mu(\rho)$ [6]

$$T_\mu(\rho) \approx \frac{(\rho/2)^{\mu-1}}{\sqrt{\pi}\,\Gamma(\mu+{}^1/_2)}\left[1 + \frac{1\cdot(2\mu-1)}{\rho^2} + \frac{1\cdot 3\cdot(2\mu-1)(2\mu-3)}{\rho^4} + \ldots\right] \tag{5.12}$$

Thus, for example, from the expressions (5.10) and (5.12) it follows that the vertical displacements decrease with a velocity directly proportional to the cube of the distance from the point of application of the force.

In Fig. 1 we have represented the vertical displacements of the points at the surface of the half-space for different values of the coefficient c. Along the abscissa axis we have represented the quantity r and along the ordinates, the quantity $W = u_z G_0 / P$. The case $c = 0$ corresponds to the homogeneous half-space.

In the expressions (5.8) we now set $r = 0$, and we expand the rational function from the integrand into simple fractions, making use of the formula [5]

$$\int_0^\infty \frac{e^{-\lambda\zeta}}{\lambda+\gamma}\,d\lambda = -e^{\gamma\zeta}\operatorname{Ei}(-\gamma\zeta) \tag{5.13}$$

where $Ei\,(-\gamma\zeta)$ is the exponential integral function. As a result we obtain the formulas for the displacement of the points of the nonhomogeneous half-space lying on the line of action of the force

$$u_z\big|_{\rho=0} = \frac{cP}{4\pi G_0(1+\zeta)}\left\{\frac{5}{2}\left(\frac{1}{\zeta}-1\right) - \left(\frac{7}{4}+\sqrt{3}\right)(\sqrt{3}\zeta-1)\,e^{\gamma_1\zeta}\,\mathrm{Ei}\,(-\gamma_1\zeta) + \left(\frac{7}{4}-\sqrt{3}\right)(\sqrt{3}\zeta+1)\,e^{\gamma_2\zeta}\,\mathrm{Ei}\,(-\gamma_2\zeta)\right\}, \quad u_r\big|_{\rho=0} = u_\beta\big|_{\rho=0} = 0 \quad (5.14)$$

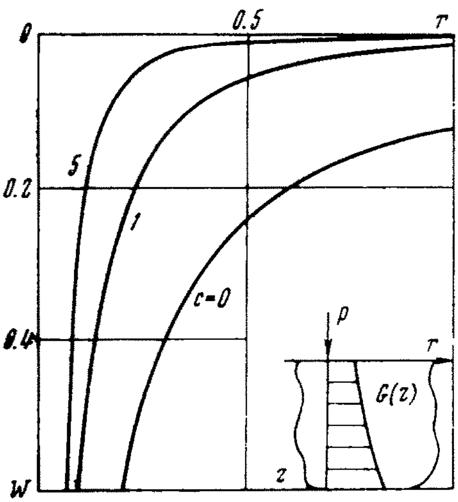

Fig. 1

6. We assume now that the concentrated force P, applied to the half-space $z \geqslant 0$ at the origin, acts in the positive direction of the x-axis tangent to the boundary surface. As in the previous problem, we will consider that the shear modulus varies with depth according to a power relation of the form (1.4) for $b = 2$, while Poisson's ratio $\nu = 1/4$. In this case the functions L and N are obtained from the expressions (5.1) and (5.2).

Thus, the determination of the state of stress and strain of a nonhomogeneous half-space reduces to the finding of three harmonic functions φ_1, φ_2 and φ_3, which allow us to satisfy the boundary conditions

$$\sigma_z\big|_{z=0} = 0, \quad \tau_{zx}\big|_{z=0} = -f(r), \quad \tau_{zy}\big|_{z=0} = 0 \qquad (6.1)$$

where $f(r)$ is the applied load, represented in the form (5.5). We seek the harmonic functions in the form

$$\varphi_1 = -\frac{\partial}{\partial x}\int_0^\infty \frac{C_1+C_2}{(c\alpha)^3} J_0(\alpha r)\,e^{-\alpha z}\,d\alpha = \cos\beta\int_0^\infty \frac{C_1+C_2}{c^3\alpha^2} J_1(\alpha r)\,e^{-\alpha z}\,d\alpha$$

$$\varphi_2 = -\frac{\partial}{\partial x}\int_0^\infty \frac{C_1-C_2}{c^2\alpha^3} J_0(\alpha r)\,e^{-\alpha z}\,d\alpha = \cos\beta\int_0^\infty \frac{C_1-C_2}{(c\alpha)^2} J_1(\alpha r)\,e^{-\alpha z}\,d\alpha \qquad (6.2)$$

$$\varphi_3 = -\frac{\partial}{\partial y}\int_0^\infty \frac{C_3}{c\alpha} J_0(\alpha r)\,e^{-\alpha z}\,d\alpha = \frac{\sin\beta}{c}\int_0^\infty C_3 J_1(\alpha r)\,e^{-\alpha z}\,d\alpha$$

Here C_1, C_2, C_3 are functions of the parameter α, subject to determination from the boundary conditions. Inserting the expressions (6.2) into (5.1) and (5.2), we have

$$L = \cos\beta\int_0^\infty \frac{e^{-\alpha z}}{\alpha^2 c} J_1(\alpha r)\{C_1\lambda(1+\zeta)[\lambda(1+\zeta)+1] + C_2[\lambda^2(1+\zeta)^2 + 3\lambda(1+\zeta)+3]\}\,d\alpha \qquad (6.3)$$

$$N = \frac{\sin\beta}{c(1+\zeta)}\int_0^\infty C_3 e^{-\alpha z} J_1(\alpha r)\,d\alpha$$

Hence, making use of the formulas (1.1) and of the relations of the theory of elasticity, it is easy to obtain the components of the displacement vector and of the stress tensor. The boundary conditions (6.1) allow us to form a system of three algebraic equations for the functions C_1, C_2 and C_3, which gives

$$C_1 = -\frac{P}{2\pi}\frac{\lambda^2+3\lambda+3}{\lambda(2\lambda^2+6\lambda+3)}, \qquad C_2 = \frac{P}{2\pi}\frac{\lambda+1}{2\lambda^2+6\lambda+3}$$

$$C_3 = \frac{P}{2\pi G_0}\frac{1}{\lambda+1} \tag{6.4}$$

Finally, we present the expressions for the determination of the displacements of the points of the nonhomogeneous half-space

$$u_r = -\frac{cP\cos\beta}{4\pi G_0(1+\zeta)}\left\{\int_0^\infty e^{-\lambda\zeta}\left[\lambda J_0(\lambda\rho) - \frac{1}{\rho}J_1(\lambda\rho)\right]\frac{2\zeta\lambda^2+3(\zeta-1)\lambda-6}{2\lambda^2+6\lambda+3}\,d\lambda - \right.$$

$$\left. \frac{2}{\rho}\int_0^\infty \frac{e^{-\lambda\zeta}}{\lambda+1}J_1(\lambda\rho)\,d\lambda\right\} \tag{6.5}$$

$$u_\beta = \frac{cP\sin\beta}{4\pi G_0(1+\zeta)}\left\{\frac{1}{\rho}\int_0^\infty e^{-\lambda\zeta}J_1(\lambda\rho)\left[\frac{2\zeta\lambda^2+3(\zeta-1)\lambda-6}{2\lambda^2+6\lambda+3} + \right.\right.$$

$$\left.\left.\frac{2}{\lambda+1}\right]d\lambda - 2\int_0^\infty \frac{\lambda e^{-\lambda\zeta}}{\lambda+1}J_0(\lambda\rho)\,d\lambda\right\}$$

$$u_z = \frac{cP\cos\beta}{4\pi G_0(1+\zeta)}\int_0^\infty e^{-\lambda\zeta}J_1(\lambda\rho)\,\lambda^2\,\frac{2\zeta\lambda+3\zeta+1}{2\lambda^2+6\lambda+3}\,d\lambda$$

In the particular case when $c\to 0$, we arrive at the well-known solution of Cerruti [7] for a concentrated force applied tangentially to the boundary surface of a homogeneous half-space. The formulas for the determination of the displacements of the points at the surface of a nonhomogeneous half-space can be obtained from (6.5) if we expand the rational expressions from the integrands into simple functions and we make use of the formulas (5.9). As a result we obtain

$$u_r\big|_{z=0} = \frac{cP\cos\beta}{2\pi G_0\rho}\left\{1-\frac{\pi}{8}\left[\frac{3}{2}\rho\left(T_0(\gamma_1\rho)+T_0(\gamma_2\rho)\right) - \gamma_2 T_1(\gamma_1\rho) - \right.\right.$$

$$\left.\left.\gamma_1 T_1(\gamma_2\rho) + 4T_1(\rho)\right]\right\} \tag{6.6}$$

$$u_\beta\big|_{z=0} = -\frac{3cP\sin\beta}{8\pi G_0\rho}\left\{1-\frac{\pi}{6}\left[\gamma_2 T_1(\gamma_1\rho)+\gamma_1 T_1(\gamma_2\rho) - 4T_1(\rho)\right] - \frac{2\pi}{3}\rho T_0(\rho)\right\}$$

$$u_z\big|_{z=0} = \frac{cP\cos\beta}{16 G_0}\left[\left(\frac{3}{2}+\sqrt{3}\right)T_1(\gamma_1\rho) + \left(\frac{3}{2}-\sqrt{3}\right)T_1(\gamma_2\rho) - \frac{6}{\pi}\right]$$

It is easy to obtain the formulas for the determination of the displacements of the points of the half-space lying on the z-axis, if we make use of the integral (5.15).

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SOME OPTIMAL PROBLEMS OF THE THEORY OF LONGITUDINAL VIBRATIONS OF RODS

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Formulations are presented of a number of optimization problems of the theory of longitudinal vibrations of rectilinear rods of constant cross section. Results of their solution, obtained by using the necessary condition of stationarity of the functional constructed in [1] and the necessary Weierstrass condition of a strong minimum of the functional established below, are described. Special attention is paid to optimization problems in which there are discontinuities in the Lagrange multipliers on the characteristic lines on equations of hyperbolic type by which longitudinal vibrations are described.

1. Formulation of the problem. Let us consider the following second order partial differential equation defined in the domain Ω $(0 \leqslant x \leqslant T,\ 0 \leqslant y \leqslant l)$:

$$z_{xx} - w^2 z_{yy} = u_1(x, y) \tag{1.1}$$

If it describes the longitudinal vibrations of a rod, then $z = z(x, y)$ is the longitudinal displacement of a rod section, and $u_1(x, y)$ is the longitudinal load intensity distributed along the rod length. Let us consider the load constrained by the inequality